

# MOTIVIC DECOMPOSITION OF PROJECTIVE PSEUDO-HOMOGENEOUS VARIETIES

SRIMATHY SRINIVASAN

**ABSTRACT.** Let  $G$  be a semi-simple algebraic group over a perfect field  $k$ . A lot of progress has been made recently in computing the Chow motives of projective  $G$ -homogeneous varieties. When  $k$  has positive characteristic, a broader class of  $G$ -homogeneous varieties appear. These are varieties over which  $G$  acts transitively with possibly non-reduced isotropy subgroup. In this paper we study these varieties which we call *projective pseudo-homogeneous varieties* for  $G$  inner type over  $k$  and prove that their motives satisfy Rost nilpotence. We also find their motivic decompositions and relate them to the motives of corresponding homogeneous varieties.

## 1. INTRODUCTION

Let  $G$  be a semi-simple algebraic group of inner type over a perfect field  $k$  of characteristic  $p > 3$ . We follow the terminology of SGA3. So by definition  $G$  is smooth and connected with trivial radical. Let  $K$  denote an algebraic closure of  $k$ .

*Definition 1.* A  $G$ -variety  $\tilde{X}$  over  $k$  is called a **projective pseudo-homogeneous variety** if  $\tilde{X}_K \simeq G_K/\tilde{P}$  for some parabolic subgroup scheme  $\tilde{P}$  in  $G_K$  that is not necessarily reduced.

Such a variety is always smooth since  $G$  is smooth (See SGA3, exp VI<sub>A</sub>, Theorem 3.2). For detailed construction of the quotient of an algebraic group by a subgroup see Chapter III, §3 of [12]. Note that by Proposition 2.1, §3, Chapter III of [12], the condition  $\tilde{X}_K \simeq G_K/\tilde{P}$  is equivalent to saying that the action map  $G(\Omega) \times \tilde{X}(\Omega) \rightarrow \tilde{X}(\Omega) \times \tilde{X}(\Omega)$  is surjective for every algebraically closed field  $\Omega$  over  $K$ . If  $\tilde{P}$  is a parabolic subgroup scheme over  $K$ , we will make slight abuse of notation and write  $G/\tilde{P}$  for  $G_K/\tilde{P}$ .

Let  $\text{Inn}(G_0)$  denote inner automorphisms of the split form  $G_0$  of  $G$ . Then  $\tilde{X}$  is a twisted form of  $G_0/\tilde{P}$  for some parabolic subgroup scheme  $\tilde{P}$  in  $G_0$ . Since  $G$  is inner,  $\tilde{X}$  corresponds to the image of a cocycle  $\sigma \in H^1(k, \text{Inn}(G_0))$  under the map  $H^1(k, \text{Inn}(G_0)) \rightarrow H^1(k, \text{Aut}^0(G_0/\tilde{P}))$  induced by the canonical map  $\text{Inn}(G_0) \rightarrow \text{Aut}^0(G_0/\tilde{P})$ . Let  $P$  denote the underlying reduced scheme of  $\tilde{P}$ . Note that since  $k$  is perfect,  $P$  is a group scheme (See §6 in Chapter VI of [30]).

*Definition 2.* Let  $\tilde{X}$  correspond to the image of the cocycle  $\sigma \in H^1(k, \text{Inn}(G_0))$  as above. We define  $X$  to be the twisted form of  $G_0/P$  obtained via the image of the same  $\sigma$  under the map  $H^1(k, \text{Inn}(G_0)) \rightarrow H^1(k, \text{Aut}^0(G_0/P))$ . Then  $X$  is a projective homogeneous variety and we say that  $X$  is the projective homogeneous variety corresponding to  $\tilde{X}$ .

By universal property of quotients, there is a canonical  $G$ -equivariant morphism  $\theta : X \rightarrow \tilde{X}$ .

*Example.* Suppose  $G = SL_3(k)$ . Let  $G/\tilde{P} \subseteq \mathbb{P}^2 \times \mathbb{P}^2$  be given by the equation  $\sum_{i=0}^2 x_i^p y_i = 0$  where the  $G$  action is  $g \cdot \vec{x} = g^{p^3} \vec{x}$  and  $g \cdot \vec{y} = (g^{-t})^{p^4} \vec{y}$  (Here by abuse of notation  $g^{p^n}$  means taking  $p^n$ th power of entries of the matrix  $g$ ). Then  $\tilde{P} = \text{Stab}([1 : 0 : 0] \times [0 : 0 : 1]) = \left\{ \begin{pmatrix} * & * & * \\ x & * & * \\ y & z & * \end{pmatrix} \mid x^{p^3} = 0, y^{p^3} = 0, z^{p^4} = 0 \right\}$ .

The underlying reduced scheme is the standard Borel  $P = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$  and the corresponding homogeneous variety  $G/P \subseteq \mathbb{P}^2 \times \mathbb{P}^2$  is given by  $\sum_{i=0}^2 x_i y_i = 0$ . This comes with the standard  $G$ -action  $g \cdot \vec{x} = g \vec{x}$  and

$g \cdot \vec{y} = (g^{-t}) \vec{y}$ . We have the canonical  $G$ -equivariant map

$$\begin{aligned} G/P &\rightarrow G/\tilde{P} \\ \vec{x} &\rightarrow \vec{x}^{p^3} \\ \vec{y} &\rightarrow \vec{y}^{p^4} \end{aligned}$$

We want to emphasize that by Theorem 5.2 in [27], the  $K$ -varieties  $G/\tilde{P}$  and  $G/P$  are not in general isomorphic. Therefore,  $X$  and  $\tilde{X}$  need not be twisted forms of each other.

In this paper we prove that Rost nilpotence theorem holds for projective pseudo-homogeneous varieties. We also compute the Chow motives of these varieties and show that their motives are isomorphic to motives of the corresponding projective homogeneous varieties.

**1.1. Notations.** Throughout this paper  $k$  is a perfect field of characteristic  $p > 3$  and  $K$  denotes the algebraic closure of  $k$ .  $\mathbb{G}_m$  denotes the usual multiplicative group.  $G$  denotes a semi-simple algebraic group of inner type over  $k$  and  $G_0$  denotes the split form of  $G$ . The set of vertices of the Dynkin diagram of  $G$  (or equivalently the set of conjugacy classes of maximal parabolics in  $G_K$ ) is denoted by  $\Delta_G$ . For a field extension  $E$  of  $k$ ,  $\tau_E \subseteq \Delta_G$  denotes the subset that contain the classes of those maximal parabolics in  $G_K$  defined over  $E$ . Given a parabolic subgroup scheme  $\tilde{P}$ ,  $P$  denotes the underlying reduced subscheme. If  $\tilde{X}$  is a projective pseudo-homogeneous variety then  $X$  denotes the corresponding projective homogeneous variety.

$\Lambda$  denotes a connected, finite, associative unital commutative ring. An example to keep in mind is a finite field of some prime characteristic. Let  $\text{Chow}(k, \Lambda)$  denote the category of Chow motives over  $k$  with coefficients in  $\Lambda$  and  $\mathcal{M}(V)$  denotes the Chow motive of  $V$ . Detailed exposition of  $\text{Chow}(k, \Lambda)$  can be found in [13]. The Tate motive  $\mathcal{M}(\text{Spec } k)(i)$  is denoted by  $\Lambda(i)$ .

**1.2. Statements of Main Results.** We say that *Krull-Schmidt principle* holds for an object in an additive category if it is isomorphic uniquely to direct sum of indecomposable summands (up to permutation). Let  $X$  be a  $k$ -variety. Recall from Karpenko's paper [24] that a summand  $M$  of  $\mathcal{M}(X)$  is called *upper* if  $Ch^0(M) \neq 0$ . See Lemma 2.8 in [24] for more details. If the motive of  $X$  satisfies Krull Schmidt principle, let  $U_X$  denote the unique upper indecomposable summand of  $\mathcal{M}(X)$ . If  $X_\tau$  is projective homogeneous corresponding to the subset  $\tau \subseteq \Delta_G$  (see §2.1), we write  $U_\tau$  for the upper indecomposable summand of  $\mathcal{M}(X_\tau)$ .

**Theorem 1.1.** (*Rost Nilpotence for Projective Pseudo-Homogeneous Varieties*) *Let  $\tilde{X}$  be a projective pseudo-homogeneous variety for a semi-simple group  $G$  of inner type over  $k$ . Then the kernel of the base change map*

$$\begin{aligned} \text{End}(\mathcal{M}(\tilde{X})) &\rightarrow \text{End}(\mathcal{M}(\tilde{X}_K)) \\ f &\mapsto f \otimes K \end{aligned}$$

*consists of nilpotents.*

*Proof.* See §4. □

**Theorem 1.2.** *The Krull-Schmidt principle holds for any shift of any summand of the motive of projective pseudo-homogeneous variety for  $G$ .*

*Proof.* This follows from Theorem 1.1, Theorem 2.3 and Theorem 5.3. □

The following theorem gives a characterization of when the motive of a variety is isomorphic to the motive a projective homogeneous variety.

**Theorem 1.3.** *Let  $X$  be projective  $G$ -homogeneous variety over  $k$ . Let  $Z$  be any geometrically split projective  $k$ -variety satisfying nilpotence principle such that the following holds in  $\text{Chow}(k, \Lambda)$ :*

- (1)  $U_X \simeq U_Z$
- (2)  $\mathcal{M}(X_L) \simeq \mathcal{M}(Z_L)$  where  $L = k(X)$

Then  $\mathcal{M}(X) \simeq \mathcal{M}(Z)$ .

*Proof.* See §6.1. □

As an application of the above theorem we derive the following main result.

**Theorem 1.4.** *Let  $G_0$  be a split semi-simple algebraic group over  $k$ . Let  $X$  and  $\tilde{X}$  be the twisted forms of  $G_0/P$  and  $G_0/\tilde{P}$  respectively corresponding to the same cocycle in  $H^1(k, \text{Inn}(G_0))$ . Then in the category of motives  $\text{Chow}(k, \Lambda)$*

$$\mathcal{M}(X) \simeq \mathcal{M}(\tilde{X})$$

*In particular, by Theorem 2.7 every indecomposable summand in  $\mathcal{M}(\tilde{X})$  is a shift of some upper motive  $U_\tau$  satisfying  $\tau_{k(X)} \subseteq \tau$ .*

*Proof.* See §6.2. □

Let  $A$  be a central simple algebra of degree  $n$  over  $k$ . Let  $X = X(d_1, d_2, \dots, d_m, A)$  be the variety of right ideals of reduced dimensions  $1 \leq d_1 < d_2 < \dots < d_m \leq n$ . Note that  $X$  is projective homogeneous for  $G = \text{PGL}(A)$ . Write  $X_K \simeq G/P$  for some parabolic subgroup  $P$ . Let  $A^{(p)} = A \otimes_{F_r} k$  and  $X^{(p)} = X \times_{F_r} \text{Spec } k$  where  $F_r : k \rightarrow k$  is the Frobenius morphism. Then it is easy to see that  $X^{(p)}_K \simeq G/\tilde{P}$  where  $\tilde{P} = G_p P$  and  $G_p$  is the kernel of the Frobenius morphism  $F_r : G \rightarrow G^{(p)}$ . Moreover,  $X$  is the projective homogenous variety corresponding to  $X^{(p)}$ .

An easy consequence of Theorem 1.4 is the following.

**Corollary 1.5.** *For a central simple algebra  $A$  over  $k$  of degree  $n$ , let  $B$  denote the central simple algebra of degree  $n$  that is Brauer equivalent to  $A^{\otimes p}$ . Then in the category  $\text{Chow}(k, \Lambda)$ , the motives of twisted flag varieties  $X(d_1, d_2, \dots, d_m, A)$  and  $X(d_1, d_2, \dots, d_m, B)$  are isomorphic. That is,*

$$\mathcal{M}(X(d_1, d_2, \dots, d_m, A)) \simeq \mathcal{M}(X(d_1, d_2, \dots, d_m, B))$$

*Taking  $m = 1$ , we get  $\mathcal{M}(SB_d(A)) \simeq \mathcal{M}(SB_d(B))$  for twisted Grassmannians. In particular, for the case of Severi-Brauer varieties we have  $\mathcal{M}(SB(A)) \simeq \mathcal{M}(SB(B))$ .*

*Proof.* Note that  $B = A^{(p)}$  by Theorem 3.9 in [25] (see also Proposition 3.2 in [14]). Therefore,

$$\begin{aligned} \mathcal{M}(X(d_1, d_2, \dots, d_m, B)) &\simeq \mathcal{M}(X(d_1, d_2, \dots, d_m, A^{(p)})) \\ &\simeq \mathcal{M}(X(d_1, d_2, \dots, d_m, A)^{(p)}) \quad (\text{by functoriality of the Frobenius}) \\ &\simeq \mathcal{M}(X(d_1, d_2, \dots, d_m, A)) \quad (\text{by Theorem 1.4}) \end{aligned}$$

The rest follows easily. □

**Remark 1.6.** Let  $A$  be a central simple algebra over  $k$  with exponent (i.e, the order of its Brauer class as an element in the Brauer group) not dividing  $p^2 - 1$ . Let  $X = SB(A)$  be the Severi-Brauer variety associated with  $A$  and let  $X^{(p)} = SB(A)^{(p)} \simeq SB(A^{(p)})$ . Then by Corollary 1.5,  $\mathcal{M}(X)$  and  $\mathcal{M}(X^{(p)})$  are isomorphic in  $\text{Chow}(k, \Lambda)$  for all coefficient rings  $\Lambda$  that are finite fields (of any characteristic). But they are not isomorphic in the integral Chow motive category  $\text{Chow}(k, \mathbb{Z})$ . Indeed, if they were isomorphic in  $\text{Chow}(k, \mathbb{Z})$ , Criterion 7.1 in [22] would imply that  $A^{(p)}$  is isomorphic either to  $A$  or its opposite  $A^{\text{op}}$ . Since  $A^{(p)}$  is Brauer equivalent to  $A^{\otimes p}$  by Proposition 3.2 in [14], this contradicts our assumption on the exponent of  $A$ . Therefore we get examples of varieties whose motives are isomorphic over all finite field coefficients but not over integral coefficients.

**1.3. Outline.** In §2 we briefly recall the facts known about projective homogeneous varieties and their motives. In §3 we give motivic decompositions of projective pseudo-homogeneous varieties for isotropic  $G$  and relate them to corresponding homogeneous varieties. In §4 we prove that Rost nilpotence holds for such varieties. In §5 we study their cellular structure and compute their motives when  $G$  is split. Finally, in §6 we compute the motivic decompositions of projective pseudo-homogeneous variety and relate them to the decompositions of corresponding homogeneous varieties.

## 2. PRELIMINARIES

Projective pseudo-homogeneous varieties are extensively studied in the literature when  $k = K$  is algebraically closed. We give a brief survey on what is known so far. In [35], Wenzel has classified all parabolic subgroup schemes  $\tilde{P}$  and in [36] he proved that the varieties  $G/\tilde{P}$  are rational. Using this classification, de Salas in [32] has classified all  $G/\tilde{P}$ . These varieties known under different names in the literature. For example, in [32] they are known as *parabolic varieties* and as *variety of unseparated flags* (VUFs) in [17]. Lauritzen and Haboush answered many interesting questions about the geometry of these varieties including canonical line bundles, vanishing theorems and Frobenius splitting in [29], [17] and [27]. Lauritzen also gave a geometric construction of  $G/\tilde{P}$  in [28] where he realizes these varieties as the  $G$ -orbit of a Borel stable line in projective space. They have rich structure and behave quite differently from the analogous *generalized flag varieties*  $G/P$  where  $P$  is smooth. For example, in [27], Lauritzen has shown that under mild assumptions on  $G$ ,  $G/\tilde{P}$  is isomorphic to  $G/P$  if and only if  $G/\tilde{P}$  is Frobenius split. In particular,  $G/P$  and  $G/\tilde{P}$  are not isomorphic in general. Moreover, in [17] one can find explicit examples of VUFs which illustrate that unlike generalized flag varieties, vanishing theorem for ample line bundles and Kodaira's vanishing theorem break down. So over algebraically closed fields, although these varieties exhibit a lot of strange phenomena, they are well understood and it is straightforward to compute their Chow motives (see §5).

However, when  $k$  is not algebraically closed, nothing much is known about them unlike the analogous projective homogeneous varieties. Projective homogeneous varieties are quite thoroughly studied in the literature ([1], [16], [13] and [26]) and so are their Chow motives ([6], [7], [9], [23] and [24]). Therefore it is natural to study projective pseudo-homogeneous varieties and ask if they exhibit any similarity to projective homogeneous varieties.

**2.1. Projective Homogeneous Varieties.** In this section we recall some facts known about projective homogeneous varieties. A  $G$ -variety  $X$  is called a *projective homogeneous variety* if  $X_K \simeq G/P$  for some parabolic subgroup  $P$  (which by definition is smooth).

The subsets of  $\Delta_G$  are in natural one-to-one correspondence with the set of conjugacy classes of parabolic subgroups in  $G_K$  defined as follows: the conjugacy class corresponding  $\tau \subseteq \Delta_G$  is the one containing the intersection of all maximal parabolics in  $\tau$  that contain a given Borel  $B$  in  $G_K$ . For any subset  $\tau \subseteq \Delta_G$ , we write  $X_\tau$  or  $X_{\tau,G}$  for the projective homogeneous variety of parabolic subgroups in  $G$  of the type  $\tau$ . For instance,  $X_{\Delta_G}$  is the variety of the Borel subgroups. Any projective  $G$ -homogeneous variety is isomorphic to  $X_\tau$  for some  $\tau$ . Let us recall some of the results known about the motives of projective homogeneous varieties.

In [6], Brosnan gave a description about the summands of the motive of projective  $G$ -homogeneous varieties for isotropic  $G$ .

**Theorem 2.1.** (Corollary 4.1 in [6]) *Let  $X$  be a projective  $G$ -homogeneous variety over  $k$ . Assume  $G$  is isotropic and let  $\lambda : \mathbb{G}_m \rightarrow G$  be an embedding of a  $k$ -split torus. Then*

$$\mathcal{M}(X) = \coprod \mathcal{M}(Z_i)(a_i)$$

where  $Z_i$  are connected components of  $X^\lambda$ . Moreover,  $Z_i$  are projective homogeneous for the centralizer  $H$  of  $\lambda$  and the twists  $a_i$  are the dimensions of the positive eigenspace of the action of  $\lambda$  on the tangent space of  $X$  at an arbitrary point  $z \in Z_i$ .

In [6], he proved that these varieties also satisfy Rost nilpotence principle. This is originally due to Chernousov, Gille and Merkurjev (Theorem 8.2 in [9]).

**Theorem 2.2.** (Theorem 5.1 in [6]) *Let  $X$  be a projective  $G$ -homogeneous variety. Then the kernel of the map*

$$\begin{aligned} \text{End}(\mathcal{M}(X)) &\rightarrow \text{End}(\mathcal{M}(X_K)) \\ f &\mapsto f \otimes K \end{aligned}$$

*consists of nilpotent endomorphisms.*

A very useful consequence of Rost nilpotence is the following result which can be found in Karpenko's paper [24].

**Theorem 2.3.** (Corollary 2.6 in [24]) *The Krull-Schmidt principle holds for any shift of any summand of the motive of any geometrically split variety in  $\text{Chow}(k, \Lambda)$  that satisfies Rost nilpotence principle.*

A very useful technique to decompose a motive is due to Rost ([31]) and Karpenko ([21]). We state this below for convenience of the reader.

**Theorem 2.4.** ([9], [10], [21]) *Let  $X$  be a smooth, projective variety over a field  $k$  with a filtration*

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \emptyset$$

*where the  $X_i$  are closed subvarieties. Assume that, for each integer  $i \in [0, n]$ , there is a smooth projective variety  $Z_i$  and an affine fibration  $\phi_i : X_i - X_{i-1} \rightarrow Z_i$  of relative dimension  $a_i$ . Then, in the category of correspondences,*

$$\mathcal{M}(X) = \coprod_{i=0}^n \mathcal{M}(Z_i)(a_i)$$

A situation where the above theorem can be applied is when  $X$  is a smooth projective variety with a  $\mathbb{G}_m$ -action. The following result is due to Iversen ([19]), Białnicki-Birula ([2], [3]) and Hesselink ([18]). See Theorem 3.3 and Theorem 3.4 in [6] for more details.

**Theorem 2.5.** ([2], [3], [18], [19]) *Let  $X$  be a smooth projective scheme over  $k$  equipped with an action of  $\mathbb{G}_m$ . Then,*

$$\mathcal{M}(X) = \coprod_i \mathcal{M}(Z_i)(a_i)$$

*where  $Z_i$  are connected components of  $X^{\mathbb{G}_m}$  and  $a_i$  are dimensions of the positive eigenspace of the action of  $\mathbb{G}_m$  on the tangent space of  $X$  at an arbitrary point in  $Z_i$ .*

Observe that any projective homogeneous variety over  $k$  is geometrically cellular i.e, has cellular decomposition (see Definition 3.2 in [20]) over the algebraic closure  $K$  and therefore by Theorem 2.4 is geometrically split i.e, its motive splits into direct sum of Tate motives over  $K$ . An important consequence of this fact, Theorem 2.2 and Theorem 2.3 is the following corollary. This is also proved by Chernousov and Merkurjev (Corollary 35 in [8]).

**Corollary 2.6.** *The Krull-Schmidt principle holds for any shift of any summand of the motive of projective homogeneous varieties in  $\text{Chow}(k, \Lambda)$ .*

The upper indecomposable motives of projective homogeneous varieties are the basic building blocks as proved by Karpenko in [24].

**Theorem 2.7.** (Theorem 3.5 in [24]) *Let  $X$  be a projective  $G$ -homogeneous variety. Then any indecomposable summand of  $\mathcal{M}(X)$  is isomorphic to  $U_\tau(i)$  for some  $i$  and some  $\tau \subseteq \Delta_G$  satisfying  $\tau_{k(X)} \subseteq \tau$ .*

### 3. MOTIVIC DECOMPOSITION FOR ISOTROPIC $G$

Recall from [19] that for a smooth projective variety  $X$  equipped with an action of  $\mathbb{G}_m$ , the fixed point locus  $X^{\mathbb{G}_m}$  is a smooth closed subscheme of  $X$ .

**Proposition 3.1.** *Let  $X$  and  $Y$  be projective varieties equipped with an action of  $\mathbb{G}_m$ . Let  $\theta : X \rightarrow Y$  be a finite surjective  $\mathbb{G}_m$ -equivariant morphism. Then the restriction morphism  $\theta|_{X^{\mathbb{G}_m}} : X^{\mathbb{G}_m} \rightarrow Y^{\mathbb{G}_m}$  is surjective.*

*Proof.* Let  $y \in Y^{\mathbb{G}_m}$  be a closed point. Clearly  $\mathbb{G}_m$  acts on the fiber  $X_y = X \times_Y \text{Spec } k(y)$ . Since  $\theta$  is finite,  $X_y$  is finite. Therefore  $\mathbb{G}_m$  fixes the underlying reduced subschemes of each point in  $X_y$ .  $\square$

A morphism  $X \rightarrow Y$  of finite type is surjective if and only if the induced map  $X(\Omega) \rightarrow Y(\Omega)$  is surjective for every algebraically closed field  $\Omega$  (EGA IV, Chapter 1, §6, Proposition 6.3.10). Using this we get an easy corollary of the above proposition.

**Corollary 3.2.** *With notations as in Proposition 3.1, let  $\{X_i\}_{i=1}^n$  and  $\{Y_i\}_{i=1}^m$  denote the connected components of  $X^{\mathbb{G}_m}$  and  $Y^{\mathbb{G}_m}$  respectively. Suppose  $\theta : X(\Omega) \rightarrow Y(\Omega)$  is bijective for every algebraically closed field  $\Omega$ . Then  $n = m$  and after permuting indices,  $\theta|_{X_i} : X_i(\Omega) \rightarrow Y_i(\Omega)$  is also bijective.*

In this section we assume that  $G$  is an isotropic, semi-simple group of inner type over  $k$ . We fix an embedding  $\lambda : \mathbb{G}_m \rightarrow G$  of a  $k$ -split torus. Let  $H$  denote the centralizer of  $\lambda$  in  $G$ . Then by Theorem 6.4.7 in [33],  $H$  is connected and semi-simple. It is defined over  $k$  by Proposition 13.3.1 of [33]. Recall that if  $X_K \simeq G/P$  and  $\tilde{X}_K \simeq G/\tilde{P}$ , we have a canonical  $G$ -equivariant morphism  $\theta : X \rightarrow \tilde{X}$ .

**Theorem 3.3.** *Let  $\tilde{X}$  be a projective pseudo-homogeneous variety for  $G$  and let  $X$  be the corresponding homogeneous variety. Then each connected component of the fixed point set  $\tilde{X}^\lambda$  is a projective pseudo-homogeneous for  $H$ . Moreover if  $\tilde{X}^\lambda = \coprod \tilde{Z}_i$ , then  $X^\lambda = \coprod Z_i$  where  $Z_i$  are the projective  $H$ -homogeneous varieties corresponding to  $\tilde{Z}_i$ .*

*Proof.* First note that  $H$  acts on  $\tilde{X}^\lambda$  because  $\lambda(t) \cdot h \cdot x = h \cdot \lambda(t) \cdot x = h \cdot x \ \forall h \in H, t \in \mathbb{G}_m, x \in \tilde{X}^\lambda$ . Let  $Y$  be a connected component of  $\tilde{X}^\lambda$ . It suffices to show that the action map  $H \times Y \rightarrow Y \times Y$  is surjective on  $\Omega$ -points for every algebraically closed field  $\Omega$  over  $K$ . By III, §1, 1.15 of [12], the  $G$ -equivariant morphism  $\theta(\Omega) : X(\Omega) \rightarrow \tilde{X}(\Omega)$  is bijective. Therefore, by Corollary 3.2,  $X^\lambda(\Omega) \rightarrow \tilde{X}^\lambda(\Omega)$  is also bijective. So there exists a connected component  $Z$  of  $X^\lambda$  such that  $\theta : Z(\Omega) \rightarrow Y(\Omega)$  is a bijection. By Theorem 7.1 in [6],  $Z$  is projective homogeneous for  $H$ . Therefore the action map  $H \times Z \rightarrow Z \times Z$  is surjective on  $\Omega$ -points. We have the following commutative diagram:

$$\begin{array}{ccc}
 H \times Z & \xrightarrow{\quad} & Z \times Z \\
 \downarrow (id, \theta) & & \downarrow (\theta, \theta) \\
 H \times Y & \xrightarrow{\quad} & Y \times Y
 \end{array}$$

The morphisms given by the top arrow and  $(\theta, \theta)$  are surjective on  $K$ -points as argued before. Hence we conclude that the bottom arrow is surjective on  $\Omega$ -points. This proves that each  $Y$  is projective

pseudo-homogeneous for  $H$ .

For the second part of the claim note that if  $x \in Z(K)$ , then  $\text{Stab}_H(x) \subseteq \text{Stab}_H(\theta(x))$ . This together with the bijectivity of  $\theta : Z(K) \rightarrow Y(K)$  shows that  $Z$  is the projective homogeneous variety corresponding to  $Y$ .  $\square$

We now analyze the action of  $\lambda$  on the tangent space at any point in the fixed point locus  $\tilde{X}^\lambda$ . As before  $X_K \simeq G/P$  and  $\tilde{X}_K \simeq G/\tilde{P}$ . Let  $b \in (G/P)^\lambda$ . Let  $a \in G/P$  be the unique point whose stabilizer in  $G_K$  is  $P$  and let  $b = g \cdot a$  for some  $g \in G(K)$ . Then  $g^{-1}\lambda g \subseteq T \subseteq P$  for some maximal torus  $T$ . Let  $T' = gTg^{-1}$ . Let  $\beta_1, \beta_2, \dots, \beta_n$  be the negative roots of  $G_K$  with respect to  $T$  and a Borel  $B$  such that  $T \subseteq B \subseteq P$ . Denote by  $\omega$ , the  $W$ -function associated to  $\tilde{P}$  as in [17] and let  $n_i = \omega(-\beta_i)$ . Without loss of generality, assume that  $\beta_1, \beta_2, \dots, \beta_m$  are the negative roots such that  $\omega(-\beta_i) < \infty$ .

**Lemma 3.4.** *With the notations above, there exists a  $T'$ -stable affine open neighborhood of  $\theta(b)$  in  $(G/\tilde{P})^\lambda$  parametrized by  $T'$ -eigen functions with weights  $p^{n_i}\alpha_i$  where  $\alpha_i$  are characters of  $T'$ . In other words, one can find an open set  $V = \text{Spec } K[X_1, X_2, \dots, X_m]$  containing  $\theta(b)$  such that*

$$t' \cdot X_i = \alpha_i^{p^{n_i}}(t') X_i \quad \forall t' \in T'$$

*Proof.* Let  $U_P^0$  denote the opposite of the unipotent radical of  $P$ . By Theorem 1 in [17],  $U = U_P^0 \cdot \theta(a) = \text{Spec } K[Y_1, Y_2, \dots, Y_m]$  is an affine open neighborhood of  $\theta(a)$  invariant under  $T$ , where

$$t \cdot Y_i = \beta_i^{p^{n_i}}(t) Y_i \quad \forall t \in T$$

Consider the affine open neighborhood  $V = gU_P^0 \cdot \theta(a)$  of  $\theta(b)$ . Then

$$T' \cdot V = T'gU_P^0 \cdot \theta(a) = gTU_P^0 \cdot \theta(a) = gU_P^0 \cdot \theta(a) = V$$

So  $V$  is  $T'$ -invariant. Moreover  $V = \text{Spec } K[X_1, X_2, \dots, X_m]$  where  $X_i = g^{-1} \cdot Y_i$ . Let  $\alpha_i$  be the character of  $T'$  defined by  $\alpha_i(t') = \beta_i(g^{-1}t'g) \quad \forall t' \in T'$ . For any point  $x \in V$ , write  $x = gy$  where  $y \in U$ . Then

$$\begin{aligned} t' \cdot X_i(x) &= t' \cdot (g^{-1} \cdot Y_i)(gy) = Y_i(g^{-1}t'gy) \\ &= \beta_i^{p^{n_i}}(g^{-1}t'g) Y_i(y) = \alpha_i^{p^{n_i}}(t') X_i(x) \quad \forall t' \in T' \end{aligned}$$

$\square$

**Lemma 3.5.** *For any point  $b \in X^\lambda$ , the dimension of positive eigenspaces of the  $\lambda$ -action on the tangent spaces at  $b$  and  $\theta(b)$  are equal.*

*Proof.* It suffices to prove the lemma over the algebraic closure  $K$  where  $X_K \simeq G/P$  and  $\tilde{X}_K \simeq G/\tilde{P}$ . So assume that  $k = K$ . By Lemma 3.4, there exists an affine open cover  $U = \text{Spec } K[Y_1, Y_2, \dots, Y_m]$  of  $b$  and an affine open cover  $V = \text{Spec } K[X_1, X_2, \dots, X_m]$  of  $\theta(b)$  parametrized by  $\lambda$ -eigen functions with weights  $\{\alpha_i\}$  and  $\{p^{n_i}\alpha_i\}$  respectively. Let  $\overline{Y_i} \in \mathfrak{m}_b/\mathfrak{m}_b^2$  and  $\overline{X_i} \in \mathfrak{m}_{\theta(b)}/\mathfrak{m}_{\theta(b)}^2$  denote the cosets of  $Y_i$  and  $X_i$  respectively. Note that  $\{\overline{Y_i}\}$  and  $\{\overline{X_i}\}$  form a basis for  $\mathfrak{m}_b/\mathfrak{m}_b^2$  and  $\mathfrak{m}_{\theta(b)}/\mathfrak{m}_{\theta(b)}^2$  respectively. It is now easy to see that the span of  $\overline{Y_i}$  is a positive eigenspace for  $\lambda$  if and only if the span of  $\overline{X_i}$  is so. By taking the dual, we are done.  $\square$

By Theorem 2.5, Theorem 3.3 and Lemma 3.5, we get the following motivic decomposition for  $\tilde{X}$ .

**Corollary 3.6.** *Let  $\tilde{X}$  and  $X$  be as in Theorem 3.3. Then*

$$\mathcal{M}(\tilde{X}) = \coprod_i \mathcal{M}(\tilde{Z}_i)(a_i)$$

and

$$\mathcal{M}(X) = \coprod_i \mathcal{M}(Z_i)(a_i)$$

where  $\tilde{Z}_i$  is projective pseudo-homogeneous for  $H$  and  $Z_i$  is the corresponding projective homogeneous variety. The twists  $a_i$  are dimensions of the positive eigenspace of the action of  $\lambda$  on the tangent space of  $X$  at an arbitrary point  $z \in Z_i$ .

Applying the above lemma inductively, we see that each of the components in the decomposition are projective (pseudo-) homogeneous for the centralizer  $Z(S)$  of a maximal  $k$ -split torus  $S$ . By Proposition 2.2 in [4], we have an almost direct product decomposition  $Z(S) = DZ(S) \cdot Z$  where  $Z$  is the center of  $Z(S)$  and  $DZ(S)$  is the semi-simple anisotropic kernel. Since the center of a group acts trivially on any projective pseudo-homogeneous variety, each of the  $\tilde{Z}_i$  (respectively  $Z_i$ ) are projective pseudo-homogeneous (respectively homogeneous) for the adjoint group of the semi-simple anisotropic kernel. Therefore we conclude:

**Corollary 3.7.** *Let  $\tilde{X}$  and  $X$  be as in Theorem 3.3. Then*

$$\mathcal{M}(\tilde{X}) = \coprod_i \mathcal{M}(\tilde{Z}_i)(a_i)$$

and

$$\mathcal{M}(X) = \coprod_i \mathcal{M}(Z_i)(a_i)$$

where each  $\tilde{Z}_i$  (respectively  $Z_i$ ) is either  $\text{Spec } k$  or anisotropic projective pseudo-homogeneous (respectively homogeneous) variety for the semi-simple anisotropic kernel of  $G$ .

*Proof.* From Corollary 3.6, each  $\tilde{Z}_i$  is projective pseudo-homogeneous variety for  $H$ . Let  $(\tilde{Z}_i)_K \simeq H/\tilde{Q}$ , for a parabolic subgroup scheme  $\tilde{Q}$  of  $H_K$ . If  $\tilde{Z}_i$  is anisotropic we are done. Suppose  $\tilde{Z}_i$  is isotropic, i.e.,  $\tilde{Z}_i$  has a  $k$ -point. Then its stabilizer is defined over  $k$  by Proposition 12.1.2 in [33]. Without loss of generality we can assume that  $\tilde{Q}$  is defined over  $k$ . Since  $k$  is perfect, the underlying reduced scheme  $Q$  is also defined over  $k$  and hence is isomorphic to  $Q(\lambda)$  for some co-character  $\lambda$  of  $H$  defined over  $k$  (Lemma 15.1.2 in [33]). So  $H$  is isotropic. If  $\lambda$  is a central torus,  $Q(\lambda) = H$  and  $\tilde{Z}_i \simeq \text{Spec } k$ . If  $\lambda$  is non-central, then we can inductively use Corollary 3.6 to get the result.  $\square$

#### 4. ROST NILPOTENCE

In this section we prove that Rost nilpotence principle holds for projective pseudo-homogeneous varieties.

*Proof of Theorem 1.1:* The proof is similar to the one in [6]. For a field extension  $L/k$ , let  $n_L$  denote the number of terms appearing in the decomposition of Corollary 3.7 for the the motive of the  $G_L$ -variety  $\tilde{X}_L$ . Clearly,  $L \subset M \Rightarrow n_M \geq n_L$  and the maximal number of terms in the coproduct occurs precisely when each  $\tilde{Z}_i$  is  $\text{Spec } L$ . In particular, this happens when  $L = K$ .

*Claim:* Set  $N(d, n) = (d+1)^{n_K - n}$  where  $d$  is the dimension of  $\tilde{X}$ . Then, for any morphism  $f \in \text{End}(M(\tilde{X}))$  with  $f \otimes K = 0$ ,  $f^{N(d, n_K)} = 0$ .

The claim obviously implies the theorem. First note that the claim is valid for  $n_k = n_K$ . Now we use descending induction on  $n = n_k$ . Let  $f \in \text{End}(M(\tilde{X}))$  be an endomorphism in the kernel of the base change map. Pick a point  $z$  in one of the anisotropic components  $\tilde{Z}_i$  in Corollary 3.7. If all components are isotropic,  $n$  is maximal and the claim is already proved. If not, set  $L = k(z)$ . Over  $L$ ,  $\tilde{Z}_i$  is isotropic. Therefore, the number  $n_i = n_L$  of terms appearing in the motivic decomposition of  $\tilde{X}_L$  is greater than  $n$ . Thus the claim holds for  $\mathcal{M}(\tilde{X}_L)$  and  $f_L^{N(d, n_i)} = 0$ . Since  $N(d, n_i) \leq N(d, n+1)$ , it follows that  $f_L^{N(d, n+1)} = 0$ . Now we use Theorem 3.1 in [5] to conclude that the composition

$$\mathcal{M}(\tilde{Z}_i)(a_i) \xrightarrow{j_1} \mathcal{M}(\tilde{X}) \xrightarrow{f^{(d+1)N(d, n+1)}} \mathcal{M}(\tilde{X})$$



vanishes where the first arrow is the canonical one coming from coproduct decomposition. Since for each summand the composition is zero, we are done.

## 5. SPLIT CASE

In this section we assume that  $G$  is split, so that  $\tilde{X} \simeq G/\tilde{P}$  and  $X \simeq G/P$ . The goal of this section is to understand the cellular structure of  $G/\tilde{P}$  and compute its motive.

**Lemma 5.1.**  *$\tilde{X}$  is a cellular variety i.e, it has decomposition into affine cells. Moreover, the affine cells can be obtained by the image of the Schubert cells in  $G/P$  under  $\theta : G/P \rightarrow G/\tilde{P}$ .*

*Proof.* We follow the proof of §2.2 in [29]. We know that  $X = G/P$  is cellular because  $G/P$  is a disjoint union of Schubert cells  $C(w) = UwP/P$  where  $U$  is the unipotent radical of  $B$ . Let  $X(w) = \overline{C(w)}$  be the corresponding Schubert variety. Let  $\tilde{X}(w)$  be the scheme theoretic image of  $X(w)$  in  $G/\tilde{P}$  under the canonical map  $\theta : G/P \rightarrow G/\tilde{P}$ . Call it a Schubert variety in  $\tilde{X}$ . We get a filtration  $\tilde{X}_0 \subseteq \tilde{X}_1 \subseteq \dots \tilde{X}_n = \tilde{X}$  of Schubert varieties where  $\tilde{X}_i - \tilde{X}_{i-1} = \coprod \theta(C(w))$ . Each of these components are homogeneous for  $U$  and hence affine by IV §4 Corollary 3.16 in [12]. So  $\tilde{X}$  is a disjoint of affine cells  $\theta(C(w))$ .  $\square$

**Lemma 5.2.** *With the notations in the proof of Lemma 5.1, the Schubert varieties  $\tilde{X}(w)$  form a basis for the Chow group of  $G/\tilde{P}$ . As a consequence  $Ch_i(G/\tilde{P}) \simeq Ch_i(G/P)$ .*

*Proof.* This follows from the previous lemma using the fact that  $\theta$  is a homeomorphism and by Example 1.9.1 in [15].  $\square$

**Theorem 5.3.** *The motive  $\mathcal{M}(\tilde{X})$  is split i.e, it decomposes into direct sum of Tate motives. Moreover,  $\mathcal{M}(X) \simeq \mathcal{M}(\tilde{X})$ .*

*Proof.* This follows directly from Corollary 3.7. Alternatively, one can also argue as follows. The fact that  $\mathcal{M}(\tilde{X})$  splits into Tate motives follows by Lemma 5.1, and Theorem 2.4. Now observe that for any variety whose motive splits into Tate motives, the rank of the  $i^{\text{th}}$  Chow group is equal to the number of summands isomorphic to  $\Lambda(i)$ . Therefore by Lemma 5.2,  $\mathcal{M}(X) \simeq \mathcal{M}(\tilde{X})$ .  $\square$

## 6. NON-SPLIT CASE

Now we remove the assumption that  $G$  is split but keep the assumption that it is inner over  $k$ . In this section we show that, the motive of any projective pseudo-homogeneous variety for  $G$  is same as the corresponding homogeneous variety. Recall the following well know fact about parabolic subgroups ([34]).

**Fact 6.1.** *Let  $G$  be a semi-simple algebraic group over a field  $k$ . Let  $P$  be a parabolic subgroup corresponding to subset  $\tau$  of nodes of the Dynkin diagram (See §2.1). Let  $\mathcal{P}$  denote the conjugacy class of  $P$ . Then  $\mathcal{P}$  contains a parabolic subgroup defined over  $k$  if and only if the nodes in  $\tau$  are circled in the Tits index of  $G$  over  $k$  and  $\tau$  is invariant under the  $*$ -action of  $\text{Gal}(K/k)$ .*

In our case, since  $G$  is assumed to be inner over  $k$ , the  $*$ -action is trivial. Let  $X$  and  $\tilde{X}$  be as before.

**Lemma 6.2.** *Let  $F$  be any field extension of  $k$ . Then  $X$  has an  $F$ -point iff  $\tilde{X}$  has a  $F$ -point.*

*Proof.* Clearly if  $X$  has an  $F$ -point, its image via the canonical map  $X \rightarrow \tilde{X}$  gives an  $F$ -point on  $\tilde{X}$ . Now assume that  $\tilde{X}$  has an  $F$ -point. Let  $F'$  be the perfect closure of  $F$ . Then by Proposition 12.1.2 of [33] the stabilizer in  $G$  of this  $F$ -point is defined over  $F'$ . Without loss of generality we can assume that  $\tilde{P}$  is defined over  $F'$ . Since  $F'$  is perfect the underlying reduced subscheme  $P$  is also defined over  $F'$ . Let  $\tau$  be the subset of nodes of Dynkin diagram corresponding to  $P$ . Since  $G$  is inner over  $k$ , it is inner over  $F$ . Therefore the  $*$ -action is trivial over  $F$ . Moreover, by Exercise 13.2.5 (4) in [33], the Tits index

of  $F'$  and  $F$  are the same. Therefore by Fact 6.1, the conjugacy class  $\mathcal{P}$  of  $P$  contains an  $F$ -defined parabolic and therefore  $X$  has an  $F$ -point.  $\square$

Note that by Theorem 1.2, the motive  $\mathcal{M}(\tilde{X})$  satisfies the Krull-Schmidt principle. Therefore we can talk about *the unique* upper summand  $U_{\tilde{X}}$  of  $\mathcal{M}(\tilde{X})$ .

**Corollary 6.3.** *Let  $X$  and  $\tilde{X}$  be as above. Then in  $\text{Chow}(k, \Lambda)$ ,  $U_X \simeq U_{\tilde{X}}$ .*

*Proof.* By Corollary 2.15 of [24], it suffices to show multiplicity one correspondences  $\alpha : \mathcal{M}(X) \rightarrow \mathcal{M}(\tilde{X})$  and  $\beta : \mathcal{M}(\tilde{X}) \rightarrow \mathcal{M}(X)$ . Take  $\alpha$  to be the correspondence induced from the canonical map  $X \rightarrow \tilde{X}$ . For  $\beta$ , first observe that  $\tilde{X}$  has an  $k(\tilde{X})$ -point. Then by Lemma 6.2, so does  $X$ . Now take  $\beta$  to be the correspondence induced from the rational map  $\tilde{X} \dashrightarrow X$ .  $\square$

*Notation:* For a variety  $X$ ,  $A^i(X, \Lambda)$  denotes the  $i^{\text{th}}$  Chow group of  $X$  with coefficients in  $\Lambda$  graded by codimension. We simply write  $A^i$  if  $X$  and  $\Lambda$  are clear from the context.  $A^{\geq i}$  denotes  $\bigoplus_{j \geq i} A^j$ . Similarly define  $A^{>i}$ ,  $A^{\leq i}$  and  $A^{<i}$ .

$A_i(X, \Lambda)$  denotes the  $i^{\text{th}}$  Chow group of  $X$  with coefficients in  $\Lambda$  graded by dimension. We make similar definitions for  $A_{\geq i}$ ,  $A_{>i}$ ,  $A_{\leq i}$  and  $A_{<i}$ .

*Definition:* Let  $\epsilon$  be the function on the objects of  $\text{Chow}(k, \Lambda)$  defined as follows:

$$\begin{aligned} \epsilon : \text{Ob}(\text{Chow}(k, \Lambda)) &\longrightarrow \mathbb{N} \cup \{-\infty\} \\ M &\longmapsto \min\{i \mid \text{Ch}^i(M) \neq 0\} \end{aligned}$$

**6.1. Proof of Theorem 1.3.** Since  $X$  is projective homogeneous variety for  $G$ , by Theorem 1.1 of [23], every indecomposable summand  $M$  of  $\mathcal{M}(X)$  is isomorphic to  $U_Y(i)$  for some projective homogeneous variety  $Y$  corresponding to  $\tau$  such that  $\tau \supseteq \tau_L$ . By condition (2),  $U_{Y_L}(i)$  comes from an indecomposable summand  $\widetilde{M}$  of  $\mathcal{M}(Z)$ . We claim that  $M \simeq \widetilde{M}$ . The claim obviously implies the theorem.

The proof of the claim is by induction on  $\epsilon(M)$ . For the base case  $\epsilon(M) = 0$ , the claim clearly holds by condition (1). Now let  $M \simeq U_Y(i)$  be a summand of  $\mathcal{M}(X)$  as above. Then  $\epsilon(M) = i$  and assume that for all indecomposable summands  $N$  with  $\epsilon(N) < i$ ,  $N \simeq \widetilde{N}$ . Write  $\mathcal{M}(X) = P \oplus Q$  where  $\epsilon(P') < i$  for every indecomposable summand  $P'$  of  $P$  and  $\epsilon(Q) \geq i$ . Then by induction hypothesis,  $\mathcal{M}(Z) \simeq P \oplus R$  where  $Q_L \simeq R_L$ . By assumption  $M$  is a summand of  $Q$  and so  $\widetilde{M}$  is a summand of  $R$ . Observe that  $\epsilon(\widetilde{M}_L) = i$  as  $\epsilon(Q_L) \geq i$ . Therefore if  $\pi \in \text{End}(\mathcal{M}(Z))$  is the projector giving rise to the summand  $\widetilde{M}$ , then  $\pi_{\overline{L}} = \sum b_k \times a_k \in \sum_r A^r \times A_r$  for  $r \geq i$  and  $a_k \cdot b_j = \delta_{kj}$ .

To complete the proof, it suffices to find  $\alpha : \mathcal{M}(Y)(i) \rightarrow \widetilde{M}$  and  $\beta : \widetilde{M} \rightarrow \mathcal{M}(Y)(i)$  such that  $\text{mult}(\beta \circ \alpha) = 1$  (See Lemma 2.14 of [24]).

For a motive  $N$  over  $k$ , let  $\overline{N}$  denote the motive base changed to  $\overline{L}$  and for a variety  $V$  over  $k$ ,  $\overline{V}$  denotes  $V \times_{\text{Spec } k} \overline{L}$ .

First note that we have  $a \in \text{Hom}(\Lambda(i), \mathcal{M}(\overline{Z})) = A_i(\overline{Z})$  given by  $\Lambda(i) \hookrightarrow \overline{U}_{Y_L}(i) \hookrightarrow \mathcal{M}(\overline{Z})$  and  $b \in \text{Hom}(\mathcal{M}(\overline{Z}), \Lambda(i)) = A^i(\overline{Z})$  given by  $\mathcal{M}(\overline{Z}) \rightarrow \overline{U}_{Y_L}(i) \rightarrow \Lambda(i)$  such that  $\text{mult}(b \circ a) = 1$  i.e.,  $a \cdot b = 1$ . Observe that with this notation,  $\overline{\pi} = b \times a + \sum_k b_k \times a_k$  where  $b_k \times a_k \in A^{\geq i} \times A_{\geq i}$ ,  $a \cdot b_k = 0 \forall b_k$  and  $a_k \cdot b = 0 \forall a_k$ .

Construction of  $\alpha$ :

Let  $\alpha_1 \in \text{Hom}(\mathcal{M}(Y_L)(i), \mathcal{M}(Z_L)) = A^{\dim Z - i}(Y_L \times Z_L)$  be given by  $\mathcal{M}(Y_L)(i) \rightarrow U_{Y_L}(i) \hookrightarrow \mathcal{M}(X_L) \xrightarrow{\sim} \mathcal{M}(Z_L)$ . Then

$$\overline{\alpha}_1 \in 1 \times a + A^{>0} \times A_{>i}$$

Let  $\alpha_2$  be the image of  $\alpha_1$  under the pull back of Chow groups

$$A^{\dim Z-i}(Y_L \times Z_L) \longrightarrow A^{\dim Z-i}(\operatorname{Spec} L(Y) \times Z)$$

induced by  $\operatorname{Spec} L(Y) \times_L Z_L \rightarrow Y_L \times_L Z_L \simeq (Y \times Z)_L$ . Then

$$\overline{\alpha}_2 = \operatorname{Spec} \overline{L}(Y) \times a.$$

Since  $\tau_L \subseteq \tau$ ,  $X$  has an  $k(Y)$ -point. So  $k(Y)(X)/k(Y) = L(Y)/k(Y)$  is purely transcendental. Therefore  $\alpha_2$  is  $k(Y)$  rational. So  $\alpha_2 \in A^{\dim Z-i}(\operatorname{Spec} k(Y) \times Z)$ . Let  $\alpha'$  be the inverse image of  $\alpha_2$  under the pull back of Chow groups

$$A^{\dim Z-i}(Y \times Z) \twoheadrightarrow A^{\dim Z-i}(\operatorname{Spec} k(Y) \times Z)$$

induced by  $\operatorname{Spec} k(Y) \times Z \rightarrow Y \times Z$ . Then

$$\overline{\alpha'} \in 1 \times a + A^{>0} \times A_{>i}$$

Let  $p: \mathcal{M}(Z) \rightarrow \widetilde{M}$  be the canonical projection. Define

$$\alpha = p \circ \alpha'$$

Construction of  $\beta$ :

Let  $\beta_1 \in \operatorname{Hom}(\mathcal{M}(Z_L), \mathcal{M}(Y_L)(i))$  be given by  $\mathcal{M}(Z_L) \xrightarrow{\sim} \mathcal{M}(X_L) \rightarrow U_{Y_L}(i) \rightarrow \mathcal{M}(Y_L)(i)$ . Then,

$$\overline{\beta}_1 \in b \times y + A^{>i} \times A_{>0}$$

where  $y$  is the class of a point in  $\overline{Y}$ . Let  $\beta_2$  be an element in the inverse image of  $\beta_1$  under the pull back of Chow groups

$$A^{\dim Y+i}(Z \times X \times Y) \rightarrow A^{\dim Y+i}(Z_L \times Y_L)$$

induced by  $Z_L \times_L Y_L \simeq (Z \times_k Y) \times \operatorname{Spec} k(X) \rightarrow Z \times Y \times X \rightarrow Z \times X \times Y$  where the last map is obtained by switching second and third factors. Then

$$\overline{\beta}_2 \in b \times 1 \times y + A^{>i} \times 1 \times A_{>0} + A^* \times A^{>0} \times A^*$$

Recall that  $\pi \in \operatorname{End}(\mathcal{M}(Z))$  is the projector giving the summand  $\widetilde{M}$ . Let  $\beta_3 = \beta_2 \circ \pi$  where  $\beta_2$  is though of as an element in  $\operatorname{Hom}(\mathcal{M}(Z), \mathcal{M}(X \times Y)(i - \dim X))$ . Then

$$\overline{\beta}_3 \in p_{134*}[(b \times a \times 1 \times 1 + \sum_k b_k \times a_k \times 1 \times 1) \cdot (1 \times b \times 1 \times y + 1 \times A^{>i} \times 1 \times A_{>0} + 1 \times A^* \times A^{>0} \times A^*)]$$

$$\overline{\beta}_3 \in b \times 1 \times y + A^i \times A^{>0} \times A^* + A^{>i} \times 1 \times A_{>0}$$

By condition (1) in the hypothesis of the theorem  $U_X \simeq U_Z$ . This implies by Corollary 2.15 of [24] that we have a multiplicity 1 correspondence  $\Gamma \in A_{\dim Z}(Z \times X)$ . Then  $\overline{\Gamma} = 1 \times x + A^{>0} \times A_{>0}$  where  $x$  refers to the class of a point in  $\overline{X}$ .

Now let  $\beta'$  be the pull back of  $\beta_3$  via  $\Gamma \times 1 \in A_{\dim Z + \dim Y}(Z \times X \times Y)$  i.e.,  $\beta' = p_{13*}[(\Gamma \times 1) \cdot \beta_3] \in A^{\dim Y+i}(Z \times Y) = \operatorname{Hom}(\mathcal{M}(Z), \mathcal{M}(Y)(i))$ . Then

$$\overline{\beta'} \in p_{13*}[(1 \times x \times 1 + A^{>0} \times A_{>0} \times 1) \cdot (b \times 1 \times y + A^i \times A^{>0} \times A^* + A^{>i} \times 1 \times A_{>0})]$$

$$\overline{\beta'} \in b \times y + A^{>i} \times A_{>0}$$

Now define  $\beta = \beta' \circ q$  where  $q: \widetilde{M} \hookrightarrow \mathcal{M}(Z)$  is the canonical map.

We now see that  $\beta \circ \alpha = \beta' \circ q \circ p \circ \alpha' = \beta' \circ \pi \circ \alpha'$ . Note that

$$\overline{\pi} \circ \overline{\alpha'} \in p_{13*}[(1 \times a \times 1 + A^{>0} \times A_{>i} \times 1) \cdot (1 \times b \times a + \sum_k 1 \times b_k \times a_k)]$$

$$\overline{\pi} \circ \overline{\alpha'} \in 1 \times a + A^{>0} \times A_{>i}$$

Finally we see that

$$\overline{\beta} \circ \overline{\alpha} \in p_{13*}[(1 \times a \times 1 + A^{>0} \times A_{>i} \times 1) \cdot (1 \times b \times y + 1 \times A^{>i} \times A_{>0})]$$

$$\overline{\beta} \circ \overline{\alpha} \in 1 \times y + A^{>0} \times A_{>0}$$

Therefore,  $\text{mult}(\beta \circ \alpha) = 1$ .

We are now ready to prove Theorem 1.4.

**6.2. Proof of Theorem 1.4.** We will prove by induction on  $n = \text{rank}(G)$ . The claim is trivially true for  $n = 0$ . Assume that the claim is true for all groups with rank less than  $n$ . Let  $\text{rank}(G) = n$ . We can assume that  $X \neq \text{Spec}(k)$  (otherwise there is nothing to prove). Let  $L = k(X)$  and  $G'$  the anisotropic kernel of  $G_L$ . Then  $\text{rank}(G') < \text{rank}(G)$ . Now by Corollary 3.7,  $\mathcal{M}(\tilde{X}_L) = \coprod_i \mathcal{M}(\tilde{Z}_i)(a_i)$  and  $\mathcal{M}(X_L) = \coprod_i \mathcal{M}(Z_i)(a_i)$  where  $\tilde{Z}_i$  is pseudo-homogeneous for  $G'$  and  $Z_i$  the corresponding homogeneous variety. By induction we have  $\mathcal{M}(\tilde{Z}_i) \simeq \mathcal{M}(Z_i)$  and thus  $\mathcal{M}(\tilde{X}_L) \simeq \mathcal{M}(X_L)$ . Moreover by Corollary 6.3  $U_X \simeq U_{\tilde{X}}$ . Therefore, by Theorem 1.3, we are done.

#### ACKNOWLEDGEMENTS

I would like to thank my advisor, Patrick Brosnan, for introducing this topic to me and suggesting this problem for my thesis. I am very grateful to him for all the useful discussions and for his feedback as it greatly improved the quality of this manuscript.

#### REFERENCES

- [1] M. Artin. Brauer-Severi varieties. In *Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981)*, volume 917 of *Lecture Notes in Math.*, pages 194–210. Springer, Berlin-New York, 1982.
- [2] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.
- [3] A. Białynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(9):667–674, 1976.
- [4] Armand Borel and Jacques Tits. Compléments à l'article: “Groupes réductifs”. *Inst. Hautes Études Sci. Publ. Math.*, (41):253–276, 1972.
- [5] Patrick Brosnan. A short proof of Rost nilpotence via refined correspondences. *Doc. Math.*, 8:69–78, 2003.
- [6] Patrick Brosnan. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math.*, 161(1):91–111, 2005.
- [7] B. Calmès, V. Petrov, N. Semenov, and K. Zainoulline. Chow motives of twisted flag varieties. *Compos. Math.*, 142(4):1063–1080, 2006.
- [8] V. Chernousov and A. Merkurjev. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups*, 11(3):371–386, 2006.
- [9] Vladimir Chernousov, Stefan Gille, and Alexander Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J.*, 126(1):137–159, 2005.
- [10] Sebastian del Baño. On the Chow motive of some moduli spaces. *J. Reine Angew. Math.*, 532:105–132, 2001.
- [11] M. Demazure. Automorphismes et déformations des variétés de Borel. *Invent. Math.*, 39(2):179–186, 1977.
- [12] Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice it Corps de classes local par Michiel Hazewinkel.
- [13] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

- [14] Mathieu Florence. On the symbol length of  $p$ -algebras. *Compos. Math.*, 149(8):1353–1363, 2013.
- [15] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.
- [16] Petrov V. Semenov N. Gille, S. and Zainoulline K. Introduction to motives and algebraic cycles on projective homogeneous varieties.
- [17] William Haboush and Niels Lauritzen. Varieties of unseparated flags. In *Linear algebraic groups and their representations (Los Angeles, CA, 1992)*, volume 153 of *Contemp. Math.*, pages 35–57. Amer. Math. Soc., Providence, RI, 1993.
- [18] Wim H. Hesselink. Concentration under actions of algebraic groups. In *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980)*, volume 867 of *Lecture Notes in Math.*, pages 55–89. Springer, Berlin, 1981.
- [19] Birger Iversen. A fixed point formula for action of tori on algebraic varieties. *Invent. Math.*, 16:229–236, 1972.
- [20] Bruno Kahn. Motivic cohomology of smooth geometrically cellular varieties. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 149–174. Amer. Math. Soc., Providence, RI, 1999.
- [21] N. A. Karpenko. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz*, 12(1):3–69, 2000.
- [22] Nikita A. Karpenko. Criteria of motivic equivalence for quadratic forms and central simple algebras. *Math. Ann.*, 317(3):585–611, 2000.
- [23] Nikita A. Karpenko. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, volume 18 of *Dev. Math.*, pages 249–257. Springer, New York, 2010.
- [24] Nikita A. Karpenko. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math.*, 677:179–198, 2013.
- [25] M. A. Knus, M. Ojanguren, and D. J. Saltman. On Brauer groups in characteristic  $p$ . In *Brauer groups (Proc. Conf., Northwestern Univ., Evanston, Ill., 1975)*, pages 25–49. Lecture Notes in Math., Vol. 549. Springer, Berlin, 1976.
- [26] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
- [27] Niels Lauritzen. Splitting properties of complete homogeneous spaces. *J. Algebra*, 162(1):178–193, 1993.
- [28] Niels Lauritzen. Embeddings of homogeneous spaces in prime characteristics. *Amer. J. Math.*, 118(2):377–387, 1996.
- [29] Niels Lauritzen. Schubert cycles, differential forms and  $\mathcal{D}$ -modules on varieties of unseparated flags. *Compositio Math.*, 109(1):1–12, 1997.
- [30] J.S. Milne. Basic theory of affine group schemes. <http://www.jmilne.org/math/CourseNotes/AGS.pdf>.
- [31] Markus Rost. The motive of a Pfister form. <https://www.math.uni-bielefeld.de/~rost/motive.html>.
- [32] Carlos Sancho de Salas. Complete homogeneous varieties: structure and classification. *Trans. Amer. Math. Soc.*, 355(9):3651–3667 (electronic), 2003.
- [33] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.
- [34] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 33–62. Amer. Math. Soc., Providence, R.I., 1966, 1966.
- [35] Christian Wenzel. Classification of all parabolic subgroup-schemes of a reductive linear algebraic group over an algebraically closed field. *Trans. Amer. Math. Soc.*, 337(1):211–218, 1993.
- [36] Christian Wenzel. Rationality of  $G/P$  for a nonreduced parabolic subgroup-scheme  $P$ . *Proc. Amer. Math. Soc.*, 117(4):899–904, 1993.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20705, USA  
 E-mail address: srimathy@math.umd.edu